

# Teaching for Abstraction: Reconstructing Constructivism

Michael Mitchelmore

*Macquarie University*

<mike.mitchelmore@mq.edu.au>

Paul White

*Australian Catholic University*

<p.white@mary.acu.edu.au>

Mathematics is abstract in the sense that mathematical concepts are formed by a process of abstraction—the recognition and reification of a similarity between a set of previously unrelated objects, events, or ideas. It follows that effective mathematics teaching should promote the abstraction process. Traditional methods seldom do so. Constructivist methods theoretically encourage abstraction, but this is rarely emphasised and the result may be an inefficient teaching procedure. Three examples are given of how explicit attention to the abstraction process could lead to radically different approaches to teaching. Some principles of what we call *Teaching for Abstraction* are abstracted from these examples.

Many students have difficulty relating abstract mathematical ideas to the everyday world. Students who do well in mathematics examinations often cannot give examples of the concepts they have learnt or suggest situations where the procedures they have carried out correctly might be useful. Students who do not do so well make little sense of mathematics and are frequently reduced to rote learning.

We believe that a major reason for this state of affairs is a misunderstanding of the nature of abstraction in mathematics. The aim of this paper is to clarify the nature of abstraction and to derive a framework for teaching mathematics intended to lead to meaningful and purposeful learning.

## Mathematics is Abstract

Mathematics deals with concepts, procedures, theorems, definitions, and proofs. The *concepts* are abstract objects (e.g., points, lines) and abstract relations between such objects (e.g., parallel, intersect). *Procedures* and *theorems* are universal statements about these concepts: rules for obtaining one abstract object or relation from others (e.g., how to construct parallel lines) and general assertions about these concepts (e.g., properties of parallel lines), respectively. Finally, *definitions* are statements relating new concepts to concepts already met, and *proofs* are logical arguments linking rules or theorems deductively.

Mathematics is abstract because the constituent concepts are abstract. But what does this mean? What is an abstract concept?

## *Abstraction as a Process*

A common theme in the work of Dienes, Piaget, and Skemp earlier this century is that concepts are formed when experiences are connected to one another on the basis of their similarities (Mitchelmore & White, 1995). Skemp (1986, p. 21) summarised the theory concisely as follows:

*Abstracting* is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. ... To distinguish between abstracting as an activity and abstraction as its end-product, we shall ... call the latter a *concept*.

In the case of everyday concepts (such as “spoon”), the similarities between examples of the concept frequently relate to the objects’ general appearance and purpose. Piaget referred to the *empirical abstraction* of everyday concepts.

Elementary mathematical concepts (such as “three”) are formed through the same process of classification and abstraction, but with one important difference: The objects which are classified are not concrete objects nor even mental objects (everyday concepts), but objects or concepts which are related to one another in a consistent manner. For example, the abstract concept “three” arises from the realisation that counting particular sets of objects (one, two, three) always ends in the same number—no matter how the objects are rearranged. Piaget called the process involved in the formation of elementary logico-mathematical concepts *reflective abstraction* and emphasised its active nature: “The [mathematical] abstraction is drawn not from the object that is acted upon, but from the action itself” (Piaget, 1970, p. 16). In common with many other writers, Piaget also emphasised the fact that abstraction is constructive. Dubinsky (1991, p. 99) even claims that “Piaget seemed to feel that this constructive aspect of reflective abstraction is more important than the abstraction (or extraction) aspect”. Sfard (1991) uses the term *reification* for the final stage of the abstraction process where the learner constructs a new mental object to embody a perceived similarity.

*Generalisation* also plays an important part in the abstraction process. After the learner has already abstracted a concept from a given set of situations, he or she may realise that the concept—possibly with some modification—can be extended to a wider set of situations. Generalisation thus refers to the process of rendering a known concept *more* general. The term is also used to mean an invariant relation between concepts—in mathematics, a procedure or a theorem.

Abstraction (together with the associated generalisation) is therefore fundamentally important both to mathematics and to the learning of mathematics (White & Mitchelmore, 1999). We would therefore expect it to be one of the main aims of mathematics teaching. Is it?

### Do Current Methods of Teaching Mathematics Teach Abstraction?

We claim that none of the usual approaches to teaching mathematics explicitly teaches the abstraction process. However, some methods do promote abstraction.

#### *The ABC Method*

Traditional mathematics teaching the world over follows what we have called the ABC Method (Mitchelmore, 1999): **A**bstract concepts and procedures are taught **B**efore **C**oncrete examples and applications. The theory is that “knowledge acquired in ‘context-free’ circumstances is supposed to be available for general application in all contexts” (Lave, 1988, p. 9), but in practice this does not happen. Because the knowledge is context-free, it is *abstract-apart* (Mitchelmore & White, 1995) and cannot be easily related to familiar situations. Concepts can only be readily applied if they are *abstract-general*, that is, if they represent that which is general to a variety of situations. The ABC method proceeds in opposite direction to the abstraction process.

#### *Constructivist Approaches to Mathematics Teaching*

Constructivists, following Piaget, emphasise the need for children to construct their own understanding but rarely (unlike Piaget) talk about abstraction. Exceptions are to be found in the works of Dubinsky (e.g., 1991) and von Glasersfeld (e.g., 1995).

However, many elements of constructivist teaching could be expected to promote abstraction: The emphasis on discussion of existing knowledge or experience; the challenge of

problem solving; the admission of contrasting methods and the reconciliation of conflicting solutions; the use of small group cooperative learning—all promote reflection which could lead to the recognition of communalities and hence either the abstraction of new concepts or the generalisation of existing ones.

For example, consider the Dutch “Realistic Mathematics Education” (RME) movement. (Treffers, 1991, p. 26) describes the five basic principles of RME:

1. Mathematics learning is constructive
2. Learning proceeds over several levels over a long time period
3. Reflection plays an important role in learning
4. Learning is interactive
5. Mathematical ideas are interconnected

The general constructivist characteristics are clearly visible.

Typically, the RME approach to teaching a topic consists of three stages:

1. Develop rules of operation in several specific, familiar, everyday contexts
2. Demonstrate that the same structure is present in several such contexts
3. Formulate, symbolise and study the common structure

Treffers (1991, p. 32) calls the first step *horizontal mathematising*: “The modeling of problem situations [so] that these can be approached with mathematical means.” The second step consists of the recognition of structural similarities and the third step the construction of a new mental object. Treffers calls these steps *vertical mathematising*, which is “directed at the perceived building and expansion of knowledge within the subject system, the world of symbols.” We can recognise the three steps as together constituting the abstraction process previously described.

### *Dienes and Multiple Embodiment*

Dienes (1963) is the only example we know of an attempt to teach abstract structures by leading students to identify similarities between isomorphic structures. For example, place value concepts were abstracted from commonalities between his Multibase Arithmetic Blocks and tree diagrams. One of Dienes’ basic principles was that of *perceptual variability*: “To abstract a mathematical structure effectively, one must meet it in a number of different situations to perceive its purely structural properties” (p. 158). Dienes aimed to create a teaching environment in which children learned by reflecting on their experiences in a variety of different situations.

Dienes was not happy with the outcomes of his experiments. “We assumed ... that abstraction would arise from a multiple embodiment of the concepts to be abstracted. By this I mean that situations physically equivalent to the concept-structure to be learned would, if handled according to specific instructions leading towards the structure, result in abstracting the common structure from all the physical situations. ... But as we observed children going through the ‘abstraction exercises’, it soon became clear that the picture was far more complex than we had assumed” (1963, p. 68). For example, requests to do “something like this” in a different embodiment or to say “how they are alike” initially brought out common features instead of a common structure—but only the first time through, as if children were just not used to looking for deeper similarities. Nevertheless, he claims that “artificial exercises in forming isomorphisms could act as teaching devices to help [children] recognise similarities when they see them” (p. 85).

Our view of Dienes' experiments is that the various "concrete materials" he used to embody mathematical structures were only concrete in the sense of being constructed out of physical materials. They were not familiar objects in children's experience and were in fact already abstractions from that experience—abstractions made by the researcher and not by the children. This view is borne out by findings that children often have difficulty relating Dienes' blocks to arithmetical procedures (Boulton-Lewis, 1992).

## Teaching for Abstraction

We suggest that it would be valuable to design mathematics teaching in such a way as to explicitly promote the abstraction of crucial mathematics concepts. To be consistent with our approach to abstraction, we shall first give some examples and then draw out the similarities which we shall call *Teaching for Abstraction*.

Mathematics has a highly developed specialist vocabulary. For example:

- Point, line, plane, angle, parallel, symmetry, congruence, theorem, proof, ...
- Length, area, volume, time, measurement, ...
- Whole numbers, fractions, integers, addition, multiplication, powers, ...
- Variable, function, equation, graph, domain, range, ...
- Linear, quadratic, exponential, polynomial, logarithm, trigonometric, ...

Some of these terms (such as "trigonometric") are only classifications, but most represent concepts—objects or relations. As such, they are candidates for teaching for abstraction. We have selected three, one each from the major curriculum areas of measurement, space, and number.

### *Example 1: Area*

Like many crucial mathematical concepts, it is not possible to define area. Any definition (such as "amount of space") is quickly seen to be circular. (Just try to define "space" without using the word "area"!)

But we all know an area when we see one, that is, we have learned to identify a certain similarity between a variety of what we now recognise as area situations. Teaching area for abstraction means helping children to learn this similarity and hence to abstract the concept of "area".

The area similarity is based on two factors: (1) It is a property of a region or a set of regions. (2) The regions are assumed to be uniform in some way, an assumption which is basic to the principle of conservation of area. So children need experience of several situations involving uniform regions if they are to abstract the concept of area. Area cannot be learned simply from cutting up paper, drawing on paper, or substituting in formulae because these experiences may not be connected to children's physical experience; paper and formulae are already abstract.

Some suitable area situations would include wall painting, book covering, floor tiling, seed planting, and land sales. All these involve operations on regions, and the uniformity assumption is either reasonable (e.g., all parts of a paddock are equally grassy) or obvious (e.g., the same tiles are used all over the floor). Children could investigate such situations in their own right before even mentioning the word "area".

Eventually, the teacher could challenge children to identify what is the same across several such situations. The similarity that is identified would then be called "area", and children could

look for some other areas situations which share the same similarity. Reflection about properties which are common to all these situations would lead to some powerful, general ideas—for example, that area measurement units may not overlap or leave gaps. Abstract relations (e.g., the formula for the area of a rectangle) could then be developed which are recognised as applicable to all area situations—and limited in their applicability by the uniformity assumption. Children would then have acquired an abstract-general concept of area.

### *Example 2: Angle*

Research (summarised by Clements & Battista, 1992) has shown that children readily form concepts of corner, slope, and turn but have difficulty integrating them into a single concept of angle. Our research (Mitchelmore, 1997, 1998; Mitchelmore & White, 1998, in press) shows that, whereas most Year 2 children see the corners in furniture, walls, and street intersections as similar in an angular sense, scissors are not readily recognised as having corners until Year 4 and leaning signposts not until Year 6. Also, about one-third of Year 8 students cannot demonstrate the angles which show how corners are similar to sloping and turning objects and a similar fraction see no similarity between a turning wheel and an opening door.

The concepts of corner, slope, opening, and turn are based on superficial similarities: Corners have two (more or less) visible angle arms, slope seems to be a property of a single static line, opening is a movement of a single line about its end-point, and turning is a global movement which need not involve any obvious lines or points. To form a concept of angle, children need to recognise a deeper similarity between all these situations—namely, that they can all be represented by angles. (The astute reader will notice that, like area, it is not possible to define angle without falling into circular definitions.)

Teaching angle for abstraction would fall into two stages. In the first stage, the teacher would help children to form separate abstract-general concepts of corners, slopes, turns, and openings by examining many examples of each concept and looking for superficial similarities. Carefully planned teaching could, we believe, lead to children forming such concepts in Year 2 or 3. Once each concept is secure, it could be developed to the point where children could compare and even measure corners, slopes, turns, and openings separately. The word “angle” need not be mentioned.

In the second stage, the teacher would lead children to link the concepts of corners, turns, and openings. Children would search for similarities both between the geometrical configurations in the three cases and between the ways that corners, slopes, and turns are measured. (For example, the right angle plays a significant but different role in each context. But four right-angled corners, turns, or openings all fill the space around a point.) The similarities would be expressed in the standard angle diagram, and the word “angle” used to name the emerging concept. Properties of the concept would then be studied (e.g., the standard method of measuring angles in degrees). Because the angle concept would have been abstracted from three superficially different contexts, it would be an abstract-general concept.

The next stage would be to generalise the angle concept, first to slope and then to other contexts such as direction and rebounds. These contexts are more complex than corners, turns, and openings and need to be studied in their own right first. For example, slope involves the physical concept of horizontal so its inclusion in the angle concept must wait until the corresponding physics is understood.

In a study funded by an ACU Large Grant, individual teaching experiments are currently under way to investigate how easy it is to teach children in Years 2 and 4 to recognise angular similarities between corners, slopes, turns, and openings.

### *Example 3: Fractions*

Hunting (1995) has shown conclusively that children often form a concept of “one-half” before primary school. He writes: “To understand ‘one-half’ in a deeper sense, a child must recognise something which several different mathematical structures have in common” (p. 122). For example, they must realise that, whenever they meet halves, two of them will always be equal and together make up the whole. They can only do this by experiencing many halving situations, reflecting on their similarities, and ignoring their differences (such as when one half of an apple has a blemish and the other is clear). It would appear that students abstract the concept of one-quarter in a similar fashion a little later.

The next step seems to be the formation of concepts of one-eighth (and possibly one-sixteenth) and three-quarters (and possibly three-eighths, etc.) by operating on one-half and one-quarter in various contexts and again noticing similarities among the relations between the various fractions in the different contexts. For example, three-quarters may be abstracted from experiences of three-quarters of an hour, three-quarters of a cup, centre three-quarter at football, and so on, linked to the knowledge that three-quarters in each case comes half way between half and full. How rapidly this abstraction is made probably depends on the activities the child regularly engages in; cooking and music seem to be particularly rich contexts for the use of fractions.

Generalisation, firstly to fractions of the form  $\frac{1}{n}$  and then  $\frac{m}{n}$ , probably occurs as a result of reflection on the similarities underlying the binary fractions already abstracted. In informal research studies, my students have frequently reported cases of 10-year old children who seem to have a solid concept of one-quarter but only as a half of a half; at least, this aspect appears more important to them than the fact that there are four quarters in a whole. Similarly, many children of the same age seem to have a concept of three-quarters but not to realise that three-quarters is three times as much as one-quarter. It is a conscious awareness of such relations, we suggest, which enables children to conceive of dividing anything into any number of equal parts and then combining them to form general fractions. We do not believe that the general fraction concept is formed by abstracting the similarities between general fractions in different contexts: Fractions like  $\frac{3}{7}$  of an hour,  $\frac{3}{7}$  back at football, and  $\frac{3}{7}$  time in music simply do not occur outside of textbooks.

Understanding of fractions therefore seems to develop rather differently from understanding of angle. Children do not form developed fraction concepts in different contexts and then abstract a more general concept later. Teaching fractions for abstraction would thus proceed rather differently than teaching angles for abstraction.

The first stage would be to assist children to abstract concepts of  $\frac{1}{2}$  and  $\frac{1}{4}$  from familiar examples of these fractions—perhaps as early as Kindergarten. The next stage would be to extend children’s fraction concepts to include  $\frac{3}{4}$  and  $\frac{1}{8}$  (and possibly  $\frac{1}{16}$ ,  $\frac{3}{8}$ , etc.) by finding these fractions in familiar contexts, exploring their relations to other fractions in those contexts, and then identifying the similarities between the contexts. Their fraction concepts would be abstract-general if they are seen to apply to all fractions contexts in a similar way.

Relations such as  $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$  would be seen to be expressions of familiar, general relations rather than the results of formal calculations.

The third stage of teaching fractions for abstraction would consist of generalising the structure inherent in the familiar fractions, still in familiar contexts but with an immediate recognition that a general structure is being constructed. For example, one would ask what “one fifth” or “two thirds” could mean and then look for examples of them in familiar contexts. The next stage would be to investigate how to combine unfamiliar fractions—but still in familiar contexts. This might include, for example, finding  $\frac{2}{3}$  of  $\frac{3}{4}$  of a cup of milk in order to reduce a recipe for 3 people to one for 2 people—first practically and then through reasoned pictorial arguments. The final stage would be to search for means of operating on fractions symbolically, and would lead to the standard “four rules” for fractions.

### *Principles of Teaching for Abstraction*

The above examples illustrate the three essential principles of Teaching for Abstraction: familiarity, similarity, and reification.

*Familiarity.* Students become familiar with several examples of the concept (i.e., several contexts from which the concept will be abstracted) before teaching the concept itself. These may be objects (e.g., tiles, furniture, road maps), operations (e.g., cutting, painting, filling, ...), or abstract ideas (e.g., half, turn, rectangle, ...). The examples are discussed using the natural language peculiar to each context (e.g., corner, slope, turn, opening), not that of the concept to be abstracted (e.g., angle). However, the teacher will anticipate the abstraction to be made later (e.g., by including examples of slopes and turns in which the two arms of the abstract angle are clearly visible).

*Similarity.* The concept is taught by finding and making explicit the similarities underlying familiar examples of that concept. The similarities may be superficial (e.g., between the appearance of corners and scissors) or structural (e.g., between the way familiar fractions are related in different contexts). Whichever it is, students’ attention is directed to the critical attributes which define these similarities and which are embodied in the concept to be abstracted (e.g., the uniformity assumption underlying the area concept). The teacher then introduces the specialist language associated with the concept and uses this vocabulary to “define” the concept (in the sense of making it more definite) by showing how it relates to the similarities on which it is based

*Reification.* As students explore the concept in more detail, it becomes increasingly a mental object in its own right, detached from any specific context. Almost any use of the concept is likely to assist its reification, providing the relation between the abstract concept and familiar examples of the concept is maintained. Some possibilities:

- Find how to use the concept in practice (e.g., by estimating the area of the school on a map).
- Investigate how to operate on the abstract concept, but always relate the results to some familiar context (e.g., calculating  $\frac{2}{3}$  of  $\frac{3}{4}$  and then checking it using a diagram).
- Define and work with special cases (e.g., percentages as special fractions).
- Look for generalisations involving the concept (e.g., area formulae).

Teaching for Abstraction clearly has much in common with other constructivist approaches, and many of their principles apply equally well. One difference is that Teaching for Abstraction has no problem with the fact that much of the content of school mathematics is pre-determined. Our belief is that, instead of merely hoping that abstract mathematical ideas will develop as a result of cooperative learning, reflection on experience, and so on, a more deliberate attempt to foster the abstraction of crucial mathematical concepts would pay handsome dividends in terms of student learning and understanding.

## References

- Boulton-Lewis, G. (1992). The processing loads of young children's and teachers' representations of place value and implications for teaching. *Mathematics Education Research Journal*, 4(1), 1-23.
- Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.420-464). New York: Macmillan.
- Dienes, Z. P. (1963). *An experimental study of mathematics-learning*. London, England: Hutchinson.
- Dubinsky, E. (1991). Constructive aspects of reflective abstraction in advanced mathematics. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 160-202). Dordrecht, The Netherlands: Kluwer.
- Hunting, R. (1995). Some results and curriculum implications from recent research on young children's pre-fraction knowledge. In J. T. Mulligan & M. C. Mitchelmore (Eds.), *Children's number learning: A research monograph of MERGA/AAMT* (pp. 109-124). Adelaide: Australian Association of Mathematics Teachers.
- Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge, England: Cambridge University Press.
- Mitchelmore, M. C. (1997). Children's informal knowledge of physical angle situations. *Learning and Instruction*, 7, 1-19.
- Mitchelmore, M. C. (1998). Young students' concepts of turning and angle. *Cognition and Instruction*, 16, 265-284.
- Mitchelmore, M. C. (1999). How to teach generalisations in mathematics? In J. M. Truran & K. M. Truran (Eds.), *Making the difference* (Proceedings of the 22nd annual conference of the Mathematics Education Research Group of Australasia, pp. 373-379). Sydney: MERGA.
- Mitchelmore, M. C., & White P. (1995). Abstraction in mathematics: Conflict, resolution and application. *Mathematics Education Research Journal*, 7, 50-68.
- Mitchelmore, M. C., & White, P. (1998). Development of angle concepts: A framework for research. *Mathematics Education Research Journal*, 10(3),4-27.
- Mitchelmore, M. C., & White, P. (in press). Development of angle concepts by successive abstraction and generalisation. *Educational Studies in Mathematics*.
- Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York, NY: Columbia University Press.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Skemp, R. (1986). *The psychology of learning mathematics* (2nd ed.). Harmondsworth: Penguin.
- Treffers, A. (1992). *Didactical background of a mathematics program for primary education*. In L. Streefland (Ed.), *Realistic mathematics education in primary school* (pp. 21-56). Utrecht, The Netherlands: CD-B Press.
- Von Glasersfeld, E. (1995). Sensory experience, abstraction, and teaching. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 369-383). Hillsdale, NJ: Lawrence Erlbaum.
- White, P., & Mitchelmore, M. C. (1999, December). Learning mathematics: A new look at generalisation and abstraction. *Refereed proceedings of the combined conference of the Australian and New Zealand Associations for Research in Education, Melbourne* [On-line]. Available: <http://www.swin.edu.au/aare/99pap/whi99309.htm>